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The two-point capacitance of infinite triangular and honeycomb networks

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Abstract. The capacitance between arbitrary two sites (vertices) in infinite triangular and honeycomb networks is studied by using Green's function. Recurrence formulas for capacitance between arbitrary sites of the triangular lattice are obtained. The capacitance for the honeycomb lattice is shown to be expressed in terms of the one for the triangular lattice.

1 Introduction

In the electrical circuit theory, one of the basic and interesting problems is the determination of the effective resistance in infinite resistor networks. Several techniques have been developed to study this problem, such as superposition of current distribution [1,2], random walks [3,4], lattice Green's function [5]. The later is the most suitable approach because it can be employed for any infinite perfect lattice structure of resistors [6,7] and perturbed lattices cases [8–13].

Wu [14] considered the problem of two-point resistance for finite lattices. He obtained an expression for the effective resistance between arbitrary two nodes in terms of the eigenvalues and eigenvectors of the real symmetric Laplacian matrix associated with the lattice. Tzeng and Wu [15] later extended the formulation of reference [14] to finite impedance networks. The Laplacian matrix associated with the impedance networks is symmetric matrix and generally complex elements.

Another interesting problem in electrical circuit analysis is the computation of the two-point capacitance in infinite capacitor networks. Based on the lattice Green's function method, some recent studies on the evaluation of two-point capacitance of perfect and perturbed regular lattices were carried out in previous works [16–21]. The capacitor electrical network can systematically be treated by the Laplacian operator of the difference equations governed by Kirchhoff's first law (conservation of electric charge) and electrical charge/voltage relationship. Then the lattice Green's function corresponding to the discrete Laplacian operator can be related to the capacitance between two arbitrary nodes in an infinite capacitor network. The lattice Green's function for the triangular and honeycomb lattices was investigated by Horiguchi [22]. He showed that the lattice Green's function for the triangular lattice is expressed in terms of the complete elliptic integrals of the first and second kind, and for the honeycomb lattice is shown to be expressed in terms of the one for the triangular lattice.

In this paper we apply the lattice Green's function approach [5,16,17] to the infinite triangular and honeycomb networks, and determine the capacitance between any two nodes in the networks. Here, we use the orthogonal Cartesian coordinates system [22] (one axis is horizontal and other is vertical as shown in Figs. 1 and 2), instead of a triangle coordinate system that usually used in the triangle lattice analysis. The advantages of the orthogonal Cartesian coordinates system used in this paper are:

(i) Some recurrence formulas for capacitances are derived from that for the lattice Green's function derived by Horiguchi [22]. With the aid of that formulas one can calculate the effective capacitance between the origin and arbitrary lattice site.
(ii) It is easier to follow for triangular lattice compared to other coordinate systems.

The paper is arranged as follows. In Section 2, the two point capacitance function for an infinite triangular lattice is studied and some recurrence formulas for capacitances are obtained. In Section 3, the two point capacitance functions for an infinite honeycomb lattice are presented. A mapping between the capacitance of honeycomb network and that of triangular network is shown. A brief conclusion is given in Section 4.

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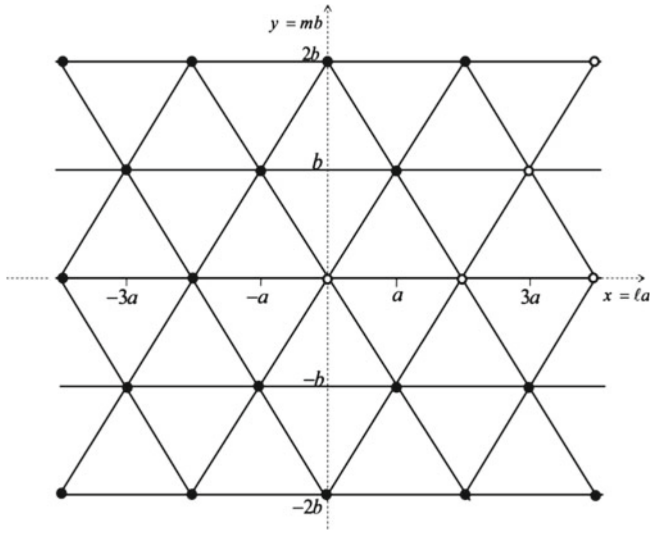

Fig. 1. The capacitor network of the triangular lattice.

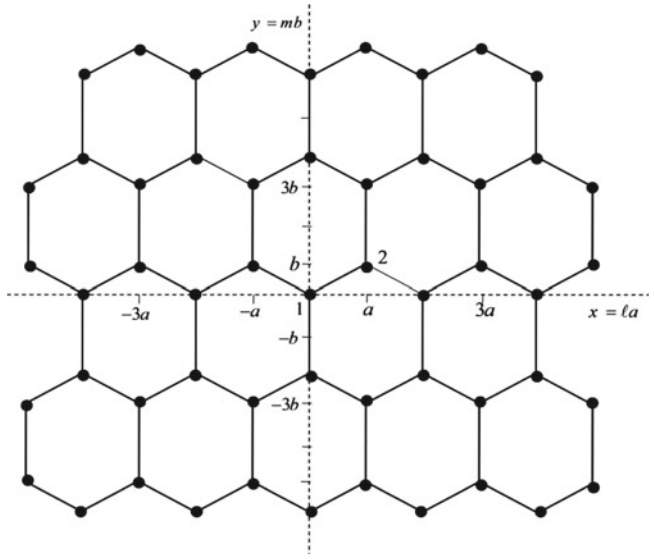

Fig. 2. The capacitor network of the honeycomb lattice.

2 Infinite triangular capacitor network

Consider an infinite triangular lattice of equal capacitances C as shown in Figure 1. Let $\mathbf{r} = \ell\mathbf{a} + m\mathbf{b}$ be the lattice site: $\ell\mathbf{a}$ is the horizontal axis and $m\mathbf{b}$ is the vertical axis, where $\ell + m$ is an even integer. If the nearest neighbor distance is chosen to be equal 1, then $a = \frac{1}{2}$ and $b = \frac{\sqrt{3}}{2}$.

Following the references [5,16,17], we evaluate the effective capacitance between the sites $\mathbf{r}_1$ and $\mathbf{r}_2$. Assume that a charge Q enters at $\mathbf{r}_1$ from a source outside the lattice and leaves at $\mathbf{r}_2$. Thus, the charge distribution at site $\mathbf{r}$ can be written as:

$$Q(\mathbf{r}) = Q(\delta(\mathbf{r}, \mathbf{r}_1) - \delta(\mathbf{r}, \mathbf{r}_2)). \tag{1}$$

Let the electric potential at site $\mathbf{r}$ is denoted by $V(\mathbf{r})$. According to Kirchhoff's first rule and electrical charge/voltage relationship, we have:

$$Q(\mathbf{r}) = \sum_{\boldsymbol{\Delta}} C(V(\mathbf{r}) - V(\mathbf{r} + \boldsymbol{\Delta})), \tag{2}$$

where $\boldsymbol{\Delta} = \pm 2\mathbf{a},\ \mathbf{a} \pm \mathbf{b}, -\mathbf{a} \pm \mathbf{b}$.

Assuming periodic boundary conditions, the potential and charge are given in terms of their Fourier transforms as:

$$V(\mathbf{r}) = \Im^{-1}[V(\mathbf{k})] = \frac{A_0}{(2\pi)^2} \int_{-\pi/a}^{\pi/a} \int_{-\pi/b}^{\pi/b} V(\mathbf{k}) \mathrm{e}^{\mathbf{ik.r}} \mathrm{d}\mathbf{k}, \tag{3}$$

$$Q(\mathbf{r}) = \Im^{-1}[Q(\mathbf{k})] = \frac{A_0}{(2\pi)^2} \int_{-\pi/a}^{\pi/a} \int_{-\pi/b}^{\pi/b} Q(\mathbf{k}) \mathrm{e}^{\mathbf{ik.r}} \mathrm{d}\mathbf{k}, \tag{4}$$

where $\mathbf{k}$ is the wavevector in the Fourier space and is limited to the first Brillouin zone [23–26] and $A_0 = ab$ is area of the unit cell, the Brillouin zone is a rectangle with sides $2\pi/a$ and $2\pi/b$ along the directions of $\mathbf{a}$ and $\mathbf{b}$, respectively, and $\mathrm{d}\mathbf{k} = \mathrm{d}k_a \mathrm{d}k_b$. Using equations (3) and (4) in (2) gives:

$$L(\mathbf{k})V(\mathbf{k}) = -Q(\mathbf{k})/C, \tag{5}$$

where $\mathbf{L}(\mathbf{k})$ is the Fourier transform of the triangular Laplacian operator $\mathbf{L}(\mathbf{r})$, given by:

$$\mathbf{L}(\mathbf{k}) = -2(3 - \cos 2\mathbf{k}\cdot\mathbf{a} - 2\cos \mathbf{k}\cdot\mathbf{a} \cos \mathbf{k}\cdot\mathbf{b}). \tag{6}$$

The lattice Green's function can be given by its Fourier transform as:

$$G(\mathbf{r}, \mathbf{r}') = \Im^{-1}[G(\mathbf{k})] = \frac{ab}{(2\pi)^2} \int_{-\pi/a}^{\pi/a} \int_{-\pi/b}^{\pi/b} G(\mathbf{k}) \mathrm{e}^{\mathbf{ik(r-r')}} \mathrm{d}k_a \mathrm{d}k_b, \tag{7}$$

where $G(\mathbf{k})$ is defined by:

$$G(\mathbf{k}) = -L^{-1}(\mathbf{k}) = \frac{1}{2(3 - \cos 2\mathbf{k}\cdot\mathbf{a} - 2\cos \mathbf{k}\cdot\mathbf{a} \cos \mathbf{k}\cdot\mathbf{b})}. \tag{8}$$

The capacitance between the sites $\mathbf{r}_1$ and $\mathbf{r}_2$ is given by the ratio:

$$C(\mathbf{r}_1, \mathbf{r}_2) = \frac{Q}{V(\mathbf{r}_1) - V(\mathbf{r}_2)}. \tag{9}$$

Using equations (1), (3), (5), (7) and (9), writing $\mathbf{r}_2 - \mathbf{r}_1 = \ell\mathbf{a} + m\mathbf{b}$ and changing the variables $\mathbf{k}\cdot\mathbf{a} = \theta_1$, $\mathbf{k}\cdot\mathbf{b} = \theta_2$, the Green's function for the triangular lattice and the capacitance between the origin and node (ℓ, m) can be obtained as:

$$G(\ell, m) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \mathrm{d}\theta_1 \times \int_{-\pi}^{\pi} \mathrm{d}\theta_2 \frac{\cos \ell\theta_1 \cos m\theta_2}{2(3 - \cos 2\theta_1 - 2\cos\theta_1 \cos\theta_2)}, \tag{10}$$

$$C(\ell,m)=\frac{C}{\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\mathrm{d}\theta_1\int_{-\pi}^{\pi}\mathrm{d}\theta_2\frac{1-\cos\ell\theta_1\cos m\theta_2}{3-\cos 2\theta_1-2\cos\theta_1\cos\theta_2}}. \tag{11}$$

From the symmetry of the network: $C(\pm1,\pm1) = C(\pm2,0)$, one can easily obtain the capacitance between nearest neighbor points, the result is $C(\pm1,\pm1) = C(\pm2,0) = 3C$.

In general the capacitance between the origin and any lattice point (ℓ,m) can be evaluated numerically from equation (11). It is shown in Appendix A, the capacitance $C(\ell,0)$ along $\ell\mathbf{a}$ axis can be calculated analytically as:

see equation (12) at the bottom of this page.

It is simple to evaluate this integral for a given ℓ. Several analytical examples are given below:

Example 1. The effective capacitance between the sites (0, 0) and (4, 0) is computed from equation (12) as:

$$C(4,0)=\frac{\pi C}{\int_0^{\pi/6}32\sin^2 y\mathrm{d}y}=\frac{3\pi C}{8\pi-12\sqrt{3}}. \tag{13}$$

Example 2. The capacitance between (0, 0) and (8, 0) is calculated to be:

$$C(8,0)=\frac{\pi C}{\int_0^{\pi/6}2(4^6\sin^6 y-4^5\sin^4 y+4^3\sin^2 y)\mathrm{d}y}=\frac{3\pi C}{928\pi-1680\sqrt{3}}, \tag{14}$$

Example 3. The capacitance $C(10,0)$ is given by:

see equation (15) at the bottom of this page.

From the lattice symmetry the capacitance is unchanged under the rotation by an angle $n\pi/3$, $n = 1, 2, 3, 4, 5$ of the coordinate axes around the origin (see Fig. 1):

$$C(\ell\mathbf{a},m\mathbf{b})=C\left(\ell\mathbf{a}\cos\frac{n\pi}{3}-m\mathbf{b}\sin\frac{n\pi}{3},\ell\mathbf{a}\sin\frac{n\pi}{3}+m\mathbf{b}\cos\frac{n\pi}{3}\right), \tag{16a}$$

and under the inversion on the $\ell\mathbf{a}$ axes and mb axes:

$$C(\ell\mathbf{a},m\mathbf{b})=C(\ell\mathbf{a},-m\mathbf{b})=C(-\ell\mathbf{a},m\mathbf{b}). \tag{16b}$$

Using equation (16a) for $n = 5$, the capacitance $C(\ell\mathbf{a},m\mathbf{b})$ for $mb > 3\ell\mathbf{a}$ can be written in terms of the capacitance for $mb < 3\ell\mathbf{a}$ as follows:

$$C(\ell\mathbf{a},m\mathbf{b})=C\left(\frac{1}{2}\ell\mathbf{a}+\frac{\sqrt{3}}{2}m\mathbf{b},\frac{\sqrt{3}}{2}\ell\mathbf{a}-\frac{1}{2}m\mathbf{b}\right). \tag{17}$$

Thus, one can only determine the capacitances between the origin and unfilled lattice points shown in Figure 1. The lattice Green's function for the triangular lattice with the nearest interaction is given by [22]:

$$G(t;\ell,m)=\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\mathrm{d}\theta_1\times\int_{-\pi}^{\pi}\mathrm{d}\theta_2\frac{\cos\ell\theta_1\cos m\theta_2}{t-\cos 2\theta_1-2\cos\theta_1\cos\theta_2}. \tag{18}$$

Comparing equation (11) with (18), we have:

$$C(\ell,m)=\frac{C}{G(3;0,0)-G(3;\ell,m)}. \tag{19}$$

In reference [22], Horiguchi showed that if t is real and $t > 3$, the lattice Green's functions $G(t,\ell,0)$ for $\ell = 0,2$ can be expressed in terms of the complete elliptic integrals of the first kind:

$$G(t;0,0)=\frac{1}{2\pi}gK(k), \tag{20a}$$

$$G(t;2,0)=\frac{t}{6\pi}gK(k)-\frac{1}{3}, \tag{20b}$$

where

$$g=\frac{8}{[(2t+3)^{1/2}-1]^{3/2}[(2t+3)^{1/2}+3]^{1/2}}, \tag{21a}$$

$$k=\frac{4(2t+3)^{1/4}}{[(2t+3)^{1/2}-1]^{3/2}[(2t+3)^{1/2}+3]^{1/2}}, \tag{21b}$$

and $K(k)$ is the complete elliptic integral of the first kind:

$$K(k)=\int_0^{\pi/2}\frac{\mathrm{d}\theta}{(1-k^2\sin^2\theta)^{1/2}}. \tag{22}$$

Substituting equations (20a) and (20b) into (19) (with $t = 3$), again the capacitance between adjacent lattice points is $C(2,0) = 3C$.

$$C(\ell,0)=\frac{\pi C}{\int_0^{\pi/6}\sum_{m=0}^{\frac{\ell}{2}-1}(-1)^{m+2}\frac{2\ell(\ell-m-1)!}{m!(\ell-2m)!}(4\sin y)^{\ell-2m-2}\sum_{n=0}^{\frac{\ell}{2}-m-1}\frac{1}{2^{2n}\sin^{2n}y}\mathrm{d}y} \tag{12}$$

$$C(10,0)=\frac{\pi C}{\int_0^{\pi/6}2(4^8\sin^8 y-(6)4^6\sin^6 y+(11)4^4\sin^4 y-(6)4^2\sin^2 y+1)dy}=\frac{3\pi C}{11249\pi-20400\sqrt{3}} \tag{15}$$

2.1 Recurrence formulas

Horiguchi [22] obtained the recurrence formulas for the Green's function $G(t;\ell,m)$ for an infinite triangular lattice. Using these results with equation (19), we obtain the following recurrence formulas for the capacitance:

$$\frac{\ell+2}{C(\ell+4,0)} = \frac{16(\ell+1)}{C(\ell+2,0)} - \frac{30\ell}{C(\ell,0)} + \frac{16(\ell-1)}{C(\ell-2,0)} - \frac{16(\ell-2)}{C(\ell-4,0)}, \tag{23a}$$

where ℓ is even and greater than or equal to 2.

$$\frac{1}{C(\ell+1,1)} = \frac{3}{C(\ell,0)} - \frac{1}{2C(\ell+2,0)} - \frac{1}{2C(\ell-2,0)} - \frac{1}{C(\ell-1,1)} \tag{23b}$$

where ℓ is even and greater than or equal to 2.

$$\frac{1}{C(\ell,m)} = \frac{6}{C(\ell-1,m-1)} - \frac{1}{C(\ell+1,m-1)} - \frac{1}{C(\ell-3,m-1)} - \frac{1}{C(\ell,m-2)} - \frac{1}{C(\ell-2,m)} - \frac{1}{C(\ell-2,m-2)} \tag{23c}$$

where $\ell \geq 4$ and is odd or even integer depend on m, and $m \geq 2$.

Using equation (17), $C(\ell,m)$ for $m > 3\ell$ can be expressed in terms of the capacitance for$m < 3\ell$:

$$C(\ell,m) = C\left(\frac{1}{2}(\ell+3m), \frac{1}{2}(\ell-m)\right). \tag{23d}$$

For $\ell = m$, the above equation becomes:

$$C(\ell,\ell) = C(2\ell,0) \quad \text{for all } \ell. \tag{23e}$$

Knowing the exact values of $C(0,0) = \infty$, $C(2,0)$ and $C(4,0)$, the two − node capacitance can be computed exactly by using the above recurrence relations. Some results are listed in Table 1.

Table 1. Capacitance $C(\ell,m)$ in units of C in infinite triangular network.

ℓ, m	$C(\ell,m)/C$	ℓ, m	$C(\ell,m)/C$
0, 0	∞	1, 1	3
2, 0	3	2, 2	2.16755
4, 0	2.16755	3, 3	1.86493
6, 0	1.86493	4, 4	1.69736
8, 0	1.69736	5, 5	1.58685
10, 0	1.58685	6, 6	1.50673
12, 0	1.50673	7, 7	1.44504
14, 0	1.44504	8, 8	1.39555
16, 0	1.39555	9, 9	1.35463
18, 0	1.35463	10, 10	1.32001
20, 0	1.32001	1, 3	1.94822
22, 0	1.29018	1, 5	1.65292
24, 0	1.26409	3, 1	2.29363
0, 2	2.29363	5, 1	1.94822
0, 4	1.77686	7, 1	1.75408
0, 6	1.56919	4, 2	1.94822
0, 8	1.44899	6, 2	1.77686
0, 10	1.36773	8, 2	1.65292
0, 12	1.30779	9, 3	1.56919
0, 14	1.26108	5, 3	1.75408
0, 16	1.22322	7, 3	1.65292

3 Infinite honeycomb capacitor network

In this section, we follow reference [5] to calculate the capacitance in an infinite honeycomb network of identical capacitances C.

The unit cell of the honeycomb network has two lattice sites labeled by $\alpha = 1,2$ as shown in Figure 2. We assume that the lattice site 1 is at the origin, and then the position of a unit cell can be specified by the position vector $\mathbf{r} = \ell\mathbf{a} + m\mathbf{b}$, where $\ell\mathbf{a}$ is the horizontal axis and $m\mathbf{b}$ is the vertical axis and $\ell + m$ is an even integer. If the nearest neighbor distance is chosen to be equal 2, then $a = \sqrt{3}$ and $b = 1$.

Let the potentials and the charges at site $\mathbf{r}$ in each unit cell are denoted by $V_\alpha(\mathbf{r})$ and $Q_\alpha(\mathbf{r})$ (with $\alpha = 1,2$), respectively. According to Kirchhoff's first rule and electrical charge/voltage relationship, the charges $Q_1(\mathbf{r})$ and $Q_2(\mathbf{r})$ at site $\mathbf{r}$ are given by:

$$Q_1(\mathbf{r}) = \sum_{\mathbf{\Delta}_1} C(V_1(\mathbf{r}) - V_2(\mathbf{r}+\mathbf{\Delta}_1)), \tag{24a}$$

$$Q_2(\mathbf{r}) = \sum_{\mathbf{\Delta}_2} C(V_2(\mathbf{r}) - V_1(\mathbf{r}+\mathbf{\Delta}_2)), \tag{24b}$$

where $\mathbf{\Delta}_1 = -2\mathbf{b}, \pm\mathbf{a}+\mathbf{b}$ and $\mathbf{\Delta}_2 = 2\mathbf{b}, \pm\mathbf{a}-\mathbf{b}$.

Again the general expressions for the inverse Fourier transforms of the potentials and charges are given by:

$$V_\alpha(\mathbf{r}) = \frac{ab}{(2\pi)^2}\int_{-\pi/a}^{\pi/a}\int_{-\pi/b}^{\pi/b} V_\alpha(\mathbf{k})e^{i\mathbf{k}.\mathbf{r}}\mathrm{d}k_a\mathrm{d}k_b, \tag{25a}$$

$$Q_\alpha(\mathbf{r}) = \frac{ab}{(2\pi)^2}\int_{-\pi/a}^{\pi/a}\int_{-\pi/b}^{\pi/b} Q_\alpha(\mathbf{k})e^{i\mathbf{k}.\mathbf{r}}\mathrm{d}k_a\mathrm{d}k_b. \tag{25b}$$

Thus, using the above equations, equations (24a) and (24b) can be written as:

$$\mathbf{L}(\mathbf{k})\begin{bmatrix} V_1(\mathbf{k}) \\ V_2(\mathbf{k}) \end{bmatrix} = -\frac{1}{C}\begin{bmatrix} Q_1(\mathbf{k}) \\ Q_2(\mathbf{k}) \end{bmatrix}, \tag{26}$$

where $\mathbf{L}(\mathbf{k})$(2 by 2 matrix) is Fourier transform of Laplacian matrix of the honeycomb network:

see equation (27) in the next page.

As usual the Green's function $\mathbf{G}(\mathbf{k})$ can be calculated by inverting $\mathbf{L}(\mathbf{k})$ to be:

see equation (28) in the next page.

$$\mathbf{L}(\mathbf{k}) = \begin{bmatrix} -3 & e^{-2i\mathbf{kb}} + e^{i(\mathbf{ka}+\mathbf{kb})} + e^{-i(\mathbf{ka}-\mathbf{kb})} \\ e^{2i\mathbf{kb}} + e^{-i(\mathbf{ka}+\mathbf{kb})} + e^{i(\mathbf{ka}-\mathbf{kb})} & -3 \end{bmatrix} \tag{27}$$

$$\mathbf{G}(\mathbf{k}) = \frac{1}{D(\mathbf{k})} \begin{bmatrix} 3 & e^{-2i\mathbf{kb}} + e^{i(\mathbf{ka}+\mathbf{kb})} + e^{-i(\mathbf{ka}-\mathbf{kb})} \\ e^{2i\mathbf{kb}} + e^{-i(\mathbf{ka}+\mathbf{kb})} + e^{i(\mathbf{ka}-\mathbf{kb})} & 3 \end{bmatrix} \tag{28}$$

$$C_{12}(\ell,m) = \frac{C}{\frac{1}{4\pi^2}\int_{-\pi}^{\pi} \mathrm{d}\theta_1 \int_{-\pi}^{\pi} \mathrm{d}\theta_2 \frac{3-\cos(\ell\theta_1+(m+2)\theta_2)-\cos((\ell-1)\theta_1+(m-1)\theta_2)-\cos((\ell+1)\theta_1+(m-1)\theta_2)}{3-\cos 2\theta_1 - 2\cos\theta_1 \cos 3\theta_2}} \tag{35}$$

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where $D(\mathbf{k}) = 2(3 - \cos 2\mathbf{ka} - 2\cos\mathbf{ka}\cos 3\mathbf{kb})$ is the determinant of matrix $\mathbf{L}(\mathbf{k})$. Thus, equation (26) becomes

$$\begin{bmatrix} V_1(\mathbf{k}) \\ V_2(\mathbf{k}) \end{bmatrix} = \frac{1}{C}\mathbf{G}(\mathbf{k}) \begin{bmatrix} Q_1(\mathbf{k}) \\ Q_2(\mathbf{k}) \end{bmatrix}. \tag{29}$$

Since the unit cell contains two lattice sites numbered by 1 and 2, there are four kinds of capacitances between any two sites $\mathbf{r}_1$ and $\mathbf{r}_2$: $C_{11}(\mathbf{r}_1,\mathbf{r}_2)$, $C_{12}(\mathbf{r}_1,\mathbf{r}_2)$, $C_{21}(\mathbf{r}_1,\mathbf{r}_2)$ and $C_{22}(\mathbf{r}_1,\mathbf{r}_2)$. From the lattice symmetry, $C_{22}(\mathbf{r}_1,\mathbf{r}_2) = C_{11}(\mathbf{r}_1,\mathbf{r}_2)$ and $C_{21}(\mathbf{r}_1,\mathbf{r}_2) = C_{12}(\mathbf{r}_2,\mathbf{r}_1)$.

To calculate the capacitance between the same kind of sites 1 and 1, $C_{11}(\mathbf{r}_1,\mathbf{r}_2)$, the charge distributions at sites 1 and 2 at $\mathbf{r}$ are:

$$Q_1(\mathbf{r}) = Q(\delta_{\mathbf{r},\mathbf{r}_1} - \delta_{\mathbf{r},\mathbf{r}_2}), \quad Q_2(\mathbf{r}) = 0. \tag{30}$$

The capacitance between the same kind of sites 1 and 1 is given by:

$$C_{11}(\mathbf{r}_1,\mathbf{r}_2) = \frac{Q}{V_1(\mathbf{r}_1) - V_1(\mathbf{r}_2)}. \tag{31}$$

Using equations (25a), (25b), (29) and (30), after writing $\mathbf{r}_2 - \mathbf{r}_1 = \ell\mathbf{a} + m\mathbf{b}$ and changing the variables $\mathbf{k}\cdot\mathbf{a} = \theta_1$, $\mathbf{k}\cdot\mathbf{b} = \theta_2$, the capacitance between the origin and node (ℓ, m) is given by:

$$C_{11}(\ell,m) = \frac{C}{\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{3(1-\cos(\ell\theta_1+m\theta_2))\mathrm{d}\theta_1\mathrm{d}\theta_2}{(3-\cos 2\theta_1 - 2\cos\theta_1\cos 3\theta_2)}}. \tag{32}$$

Now, to calculate the capacitance between the different kinds of vertices 1 and 2, $C_{12}(\mathbf{r}_1,\mathbf{r}_2)$, the charge distributions at sites 1 and 2 at $\mathbf{r}$ are:

$$Q_1(\mathbf{r}) = Q\delta_{\mathbf{r},\mathbf{r}_1}, \quad Q_2(\mathbf{r}) = -Q\delta_{\mathbf{r},\mathbf{r}_2}. \tag{33}$$

The capacitance between 1, 2-type points, $C_{12}(\mathbf{r}_1,\mathbf{r}_2)$ is given by:

$$C_{12}(\mathbf{r}_1,\mathbf{r}_2) = \frac{Q}{V_1(\mathbf{r}_1) - V_2(\mathbf{r}_2)}. \tag{34}$$

Following the same procedures of $C_{11}(\mathbf{r}_1,\mathbf{r}_2)$, the capacitance $C_{12}(\mathbf{r}_1,\mathbf{r}_2)$ can be obtained as:

see equation (35) at the top of this page.

The equivalent capacitance between first neighbors nodes (the capacitance between point 1 at (0, 0) and 2 at (0, 0)) can be easily obtained from equation (35). From symmetry of the lattice: $C_{12}(1,1) = C_{12}(-1,1) = C_{12}(0,-2)$ and using equation (35) this capacitance is $3C/2$. Also the equivalent capacitance, $C_{11}(2,0)$ between second nearest neighbors nodes can be calculated from equation (32) in similar way, this capacitance is C.

The capacitance $C_{12}(\ell,m)$ can be written in terms of $C_{11}(\ell,m)$ as:

$$\frac{1}{C_{12}(\ell,m)} = \frac{1}{3C_{11}(\ell,m+2)} + \frac{1}{3C_{11}(\ell-1,m-1)} + \frac{1}{3C_{11}(\ell+1,m-1)}. \tag{36}$$

It is well known in the literature the honeycomb lattice is the dual lattice of a triangular lattice. Therefore, the capacitances $C_{11}(\ell,m)$ and $C_{12}(\ell,m)$ for the honeycomb lattice can be expressed in terms of the capacitances for the triangular lattice:

$$C_{11}(\ell,m) = \frac{1}{3}C_{\mathrm{trai}}\left(\ell, \frac{1}{3}m\right), \tag{37}$$

$$\frac{1}{C_{12}(\ell,m)} = \frac{1}{C_{\mathrm{trai}}(\ell,\frac{1}{3}(m+2))} + \frac{1}{C_{\mathrm{trai}}(\ell+1,\frac{1}{3}(m-1))} + \frac{1}{C_{\mathrm{trai}}(\ell-1,\frac{1}{3}(m-1))}. \tag{38}$$

Table 2. Capacitances $C_{11}(\ell,m)$ and $C_{12}(\ell,m)$ in units of C in infinite honeycomb network.

ℓ, m	$C_{11}(\ell,m)/C$	ℓ, m	$C_{12}(\ell,m)/C$
0, 0	∞	0, 0	∞
2, 0	1	1, 1	3/2
4, 0	0.72252	3, 1	0.81256
6, 0	0.62164	0, 4	0.90690
8, 0	0.56579	4, 4	0.68373
10, 0	0.52895	2, 4	0.81256
12, 0	0.50224	0, −2	3/2
14, 0	0.48168	3, 1	0.81256
16, 0	0.46518	5, 1	0.66188
18, 0	0.45145	7, 1	0.58981
1, 3	1	1, 7	0.70890
3, 3	0.76454	2, −2	0.90689
6, 6	0.59229	4, −2	0.70888
2, 6	0.72252	6, −2	0.61739

For derivation these results we have substituted $\theta_2 = \theta_2'/3$ into equations (32) and (35) then we have used the following integration property:

$$\frac{1}{c}\int_{ca}^{cb} f\left(\frac{x}{c}\right)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x. \tag{39}$$

Thus, from equations (37) and (38) the two-point capacitances on the honeycomb network are obtained from the knowledge of the one on the triangular lattice. Some results are given in Table 2.

4 Conclusion

In this paper using the lattice Green's function method [5,16,17] we calculated the capacitance for the infinite triangular and honeycomb networks lattices of identical capacitors. The orthogonal Cartesian coordinates system is used instead of a triangle coordinate system. We derived recurrence relations for the capacitance of a triangular lattice. We derived explicit expressions for the capacitances between two arbitrary lattice points in honeycomb lattice in terms of the capacitances on a triangular lattice.

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Appendix A: The expression of the capacitance C(l,0) along la axis for a triangular lattice

Starting from equation (19) for the capacitance $C(\ell,0)$ along $\ell\mathbf{a}$ axis we have:

$$C(\ell,0) = \frac{C}{I_\ell}, \tag{A1}$$

where

$$I_\ell = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{(1-\cos\ell\theta_1)\mathrm{d}\theta_1\mathrm{d}\theta_2}{3-\cos 2\theta_1 - 2\cos\theta_1\cos\theta_2}, \tag{A2}$$

and $\ell \geq 2$, and an even integer.

Performing the integral over θ_2 using the residue method:

$$\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_2}{a-b\cos\theta_2} = \frac{2\pi}{\sqrt{a^2-b^2}}. \tag{A3}$$

Hence, the integral I_ℓ becomes:

$$I_\ell = \int_0^{\pi}\frac{\mathrm{d}\theta_1}{2\pi}\frac{1-\cos\ell\theta_1}{\sin\theta_1\sqrt{4-\cos^2\theta_1}}. \tag{A4}$$

The function $T_\ell(x) = \cos\ell\theta_1$ is known as the Chebyshev polynomials of type I [27], where $x = \cos\theta_1$. The power-series representation of $T_\ell(x)$ is given by:

$$T_\ell(\cos\theta_1) = \cos\ell\theta_1 = \frac{\ell}{2}\sum_{m=0}^{\ell/2}(-1)^m\frac{(\ell-m-1)!}{m!(\ell-2m)!} \times(2\cos\theta_1)^{\ell-2m}. \tag{A5}$$

$$C(\ell,0)=\frac{C}{\int_0^{\pi/6}\frac{\mathrm{d}y}{\pi}\sum_{m=0}^{\frac{\ell}{2}-1}(-1)^{m+2}\frac{2\ell(\ell-m-1)!}{m!(\ell-2m)!}(4\sin y)^{\ell-2m-2}\sum_{n=0}^{\frac{\ell}{2}-m-1}\frac{1}{2^{2n}\sin^{2n}y}} \tag{A8}$$

Let $\cos\theta_1=2\sin y$ the integral I_ℓ becomes:

$$I_\ell=\int_0^{\pi/6}\frac{\mathrm{d}y}{2\pi}\frac{\ell\sum_{m=0}^{\frac{\ell}{2}-1}(-1)^{m+1}\frac{(\ell-m-1)!}{m!(\ell-2m)!}(4\sin y)^{\ell-2m}}{1-4\sin^2 y}. \tag{A6}$$

Hence, the capacitance $C(\ell,0)$ is:

$$C(\ell,0)=\frac{C}{\int_0^{\pi/6}\frac{\mathrm{d}y}{2\pi}\frac{\ell\sum_{m=0}^{\frac{\ell}{2}-1}(-1)^{m+1}\frac{(\ell-m-1)!}{m!(\ell-2m)!}(4\sin y)^{\ell-2m}}{1-4\sin^2 y}}. \tag{A7}$$

By writing $\frac{1}{1-4\sin^2 y}=\frac{(2\sin y)^{-2}}{(2\sin y)^{-2}-1}=-x\frac{1}{1-x}$, where $x=(2\sin y)^{-2}$ and expanding $\frac{1}{1-x}$ in Taylor series about $x=0$, we have:

$$\frac{1}{1-4\sin^2 y}=(-1)^{+1}\sum_{n=0}(2\sin y)^{-2n-2}.$$

Equation (A7) can be written as:

$$C(\ell,0)=$$
$$\frac{C}{\int_0^{\pi/6}\frac{\mathrm{d}y}{\pi}\sum_{m=0}^{\frac{\ell}{2}-1}(-1)^{m+2}\frac{2\ell(\ell-m-1)!}{m!(\ell-2m)!}\sum_n 2^{2n}(4\sin y)^{\ell-2m-2n-2}}.$$

Note that m in the summation is positive integer from zero to $\frac{\ell}{2}-1$ (not the site m). The integrand in the above equation is a polynomial of $\sin y$. Therefore, the upper limit of n in the summation can be determined by:

$$\ell-2m-2n-2\geq 0 \text{ or } n\leq\frac{\ell}{2}-m-1.$$

Finally, the expression for $C(\ell,0)$ in equation (A7) can be written as:

See equation (A8) at the top of this page.